Math 250 . Briggs 2/e

7.2 Natural log and Natural Exponential Functions – day 3

Objectives.

1) Understand and use properties of the natural exponential function $f(x) = e^x$

2) Differentiate natural exponential functions

3) Integrate natural exponential functions

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Solve

x ① $\dfrac{5000}{1+e^{2x}} = 2$

$5000 = 2(1+e^{2x})$	clear fractions
$2500 = 1+e^{2x}$	divide both sides by 2
$2499 = e^{2x}$	subtract 1 from both sides
$\ln 2499 = \ln e^{2x}$	take ln both sides
$\ln 2499 = 2x$	simplify using inverse property
$\frac{1}{2}\ln 2499 = x$	isolate x by dividing both sides by 2
$\boxed{x = \ln\sqrt{2499}}$	← give exact answer

x ② Solve $\ln(x-2)^2 = 12$

$e^{\ln(x-2)^2} = e^{12}$	exponentiate both sides } OR equiv. exp. eqn.
$(x-2)^2 = e^{12}$	simplify using inverse property
$x-2 = \pm\sqrt{e^{12}}$	square root property
$x-2 = \pm e^6$	simplify $(e^{12})^{1/2}$ by multiplying exponents
$\boxed{x = 2\pm e^6}$	add 2 to both sides
	(neg is <u>not</u> extraneous)

x ③ Solve $2\ln(x-2) = 12$

$\ln(x-2) = 6$	divide both sides by 2
$e^{\ln(x-2)} = e^6$	exponentiate both sides } OR. equiv. exp. eqn.
$x-2 = e^6$	simplify inverse property
$\boxed{x = 2+e^6}$	add 2 to both sides

<u>Note</u>: ② has two solutions, while ③ has only one solution. Use the more complex argument $(x-2)^2$ to capture all possible solutions, then check for extraneous results in original equation.

e.g. $2-e^6$ in ②:

$2\ln((2-e^6)-2) = 12$

$2\ln(-e^6) = 12$

not defined. Reject $2-e^6$ as solution to ③.

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✓ (4) Solve $\ln(x+2) + \ln(x-1) = 3$

$$\ln[(x+2)(x-1)] = 3$$ combine to get more complex argume[nt]

$$e^3 = (x+2)(x-1)$$ equivalent exponential equation

$$e^3 = x^2 + x - 2$$ FOIL

$$0 = x^2 + x + (e^3 - 2)$$ set = 0

$$x = \frac{-1 \pm \sqrt{1 - 4(1)(e^3-2)}}{2}$$ solve by quadratic formula.

$$x = \frac{-1 \pm \sqrt{1 - 4e^3 + 8}}{2}$$

$$x = \frac{-1 \pm \sqrt{9 - 4e^3}}{2}$$

$x \approx 4.23$
$\quad -5.23$

Find approximate values to check for domain.

-5.23 is not in domains of $\ln(x+2)$ and $\ln(x-1)$ reject as extraneous.

$$\boxed{x = \frac{-1 + \sqrt{9 - 4e^3}}{2}}$$ ← exact answer

X (5) Illustrate that $f(x) = e^{x-1}$ and $g(x) = 1 + \ln x$ are inverse, algebraically:

$f(g(x)) = x$: $f(g(x)) = e^{1 + \ln x - 1} = e^{\ln x} = x$ ✓

$g(f(x)) = x$: $g(f(x)) = 1 + \ln(e^{x-1}) = 1 + x - 1 = x$ ✓

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Properties of the natural exponential function e^x

domain of $f(x)=e^x$ is $(-\infty,\infty)$
range of $f(x)=e^x$ is $(0,\infty)$.

$f(x)=e^x$ is increasing, continuous, one-to-one, concave up } on its entire domain.

$\lim_{x\to\infty} e^x = +\infty$ — end behavior, limit at infinity.

$\lim_{x\to -\infty} e^x = 0$ — end behavior, limit at neg. infinity is horizontal asymptote.

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* $\frac{d}{dx}[e^x] = e^x$

Proof:

$\ln e^x = x$ — inverse property

$\frac{1}{e^x}\cdot\frac{d}{dx}[e^x] = 1$ — differentiate both sides

$\frac{d}{dx}[e^x] = e^x$ — solve for $\frac{d}{dx}[e^x]$.

Consequence: If $y=e^x$ and $y'=e^x$ then $y'=y$.

$y=e^x$ is its own derivative

$y=e^x$ is a solution of the DE $y'=y$

chain rule: $\frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$.

This means also:

$$\int e^x\,dx = e^x + C$$

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⑥ Find the derivative of $f(x) = e^{3x+1}$

$$f'(x) = \underbrace{e^{3x+1}}_{\frac{d}{dx}(\text{outside})} \cdot \underbrace{3}_{\frac{d}{dx}(\text{inside})}$$

$$\boxed{f'(x) = 3e^{3x+1}}$$

Use calculus to show intervals of increasing or decreasing and concave up or down.

⑦ $f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

critical values $e^x = 0$ none

e^x undef none

f' ← + → so $\boxed{\text{increasing } (-\infty, \infty)}$

f'' ← + → so $\boxed{\text{concave up } (-\infty, \infty)}$

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X (8) Write the equation of the tangent line to $y = e^{-3x}$ at $(0,1)$.

step 1: Find derivative + evaluate to get slope of tangent line.

$$f'(x) = e^{-3x} \cdot (-3) \qquad \text{chain rule}$$

$$f'(x) = -3e^{-3x}$$

$$f'(0) = -3e^{-3(0)}$$

$$= -3e^{0}$$

$$= -3 \cdot 1$$

$$= -3 = m.$$

step 2: Substitute into point-slope formula

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -3(x - 0)$$

$$y - 1 = -3x$$

$$\boxed{3x + y - 1 = 0}$$

✓ (9) Find $\frac{dy}{dx}$ when $y = x^2 e^{-x}$

$$y' = x^2 \cdot \frac{d}{dx}[e^{-x}] + \frac{d}{dx}[x^2] \cdot e^{-x} \qquad \text{product rule}$$

$$y' = x^2 \cdot e^{-x} \cdot \frac{d}{dx}(-x) + 2xe^{-x} \qquad \text{chain rule}$$

$$y' = x^2 e^{-x} \cdot (-1) + 2xe^{-x}$$

$$y' = -x^2 e^{-x} + 2xe^{-x}$$

$$\boxed{y' = -xe^{-x}(x-2)} \qquad \text{factor least powers}$$

X (10) Find intervals where $y = x^2 e^{-x}$ is increasing and/or decreasing.

step 1: Find derivative (done in (8))

step 2: Find critical values.

$$y' = 0 \qquad \frac{-x(x-2)}{e^x} = 0$$

$$-x(x-2) = 0$$

$$x = 0, \quad x = 2$$

$y' =$ undef only if $e^x = 0$ (none)

step 3: sign chart for $y'(x)$.

(−) | (+) | (−)

0 2

step 4: Write intervals

increasing $(0, 2)$
decreasing $(-\infty, 0) \cup (2, \infty)$

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✓ (11) Find the first derivative of $f(x) = \ln\left(\frac{1+e^x}{1-e^x}\right)$

CRUCIAL Use log properties to expand first

$f(x) = \ln(1+e^x) - \ln(1-e^x)$ — log property

$f'(x) = \frac{1}{1+e^x} \cdot \frac{d}{dx}(1+e^x) - \frac{1}{1-e^x} \cdot \frac{d}{dx}(1-e^x)$ — chain rule

$$f'(x) = \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x}$$

$$= \frac{e^x(1-e^x)}{(1+e^x)(1-e^x)} + \frac{e^x(1+e^x)}{(1-e^x)(1+e^x)}$$ find common denominator

$$= \frac{e^x - e^{2x} + e^x + e^{2x}}{(1+e^x)(1-e^x)}$$

$$\boxed{f'(x) = \frac{2e^x}{(1+e^x)(1-e^x)}}$$ leave factored

Integrate

X (12) $\int e^{1-3x}\,dx$

$u = 1-3x$ — u-substitution

$du = -3dx$

$= -\frac{1}{3}\int e^{1-3x}(-3dx)$ — $\left.\begin{array}{l}\text{multiply inside by } -3 \\ \text{outside by } -\frac{1}{3}\end{array}\right\}$ net result 1

$= -\frac{1}{3}\int e^u\,du$ — rewrite

$= -\frac{1}{3}e^u + C$ — antidifferentiate

$= \boxed{-\frac{1}{3}e^{1-3x} + c}$ — Substitute back.

X (13) $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\,dx$

$u = 1+e^{2x}$
$du = e^{2x}\cdot 2\,dx$ — u-substitution

$= \frac{1}{2}\displaystyle\int \frac{1}{1+e^{2x}}\cdot 2e^{2x}\,dx$ — mult inside by 2, outside by $\frac{1}{2}$ } net result 1

$= \frac{1}{2}\displaystyle\int \frac{1}{u}\,du$ — rewrite in u.

$= \frac{1}{2}\ln|u| + C$ — antidifferentiate

$= \frac{1}{2}\ln|1+e^{2x}| + C$ — substitute back

$= \frac{1}{2}\ln(1+e^{2x}) + C$ — notice $1+e^{2x}$ is always positive, no absolute value needed,

$= \boxed{\ln\sqrt{1+e^{2x}} + C}$

✓ (14) $\displaystyle\int_{-2}^{0} x^2 e^{x^3/2}\,dx$

$u = \frac{x^3}{2}$ — u-substitution
$du = \frac{3}{2}x^2\,dx$

$u_1(-2) = \frac{(-2)^3}{2} = \frac{-8}{2} = -4$
$u_2(0) = \frac{0^3}{2} = 0$ — definite integral— convert limits of integration to u

$= \frac{2}{3}\displaystyle\int_{-2}^{0} e^{x^3/2}\cdot\frac{3}{2}x^2\,dx$ — multiply inside by $\frac{3}{2}$, outside by $\frac{2}{3}$ } net result 1

$= \frac{2}{3}\displaystyle\int_{-4}^{0} e^{u}\,du$ — rewrite in u

$= \frac{2}{3}\left[e^u\right]\Big|_{-4}^{0}$ — antidifferentiate continued

$= \frac{2}{3}\left[e^{0} - e^{-4}\right]$ evaluate at limits of integration using FTC

$= \frac{2}{3}(1 - e^{-4})$ simplify zero power

$= \frac{2}{3}\left(\frac{e^4}{e^4} - \frac{1}{e^4}\right)$ write with common denominator

$= \boxed{\frac{2}{3}\left(\frac{e^4 - 1}{e^4}\right)}$

✓ 15 $\displaystyle\int \frac{2e^{x} - 2e^{-x}}{(e^{x}+e^{-x})^2}\,dx$

$u = e^{x} + e^{-x}$

$du = \left(e^{x} + e^{-x}\cdot(-1)\right)dx$ u substitution

$= (e^{x} - e^{-x})\,dx$

$= 2\displaystyle\int \frac{1}{(e^{x}+e^{-x})^2}\cdot(e^{x} - e^{-x})\,dx$ factor out 2 using property of integrals

$= 2\displaystyle\int \frac{1}{u^2}\,du$ rewrite in u

$= 2\displaystyle\int u^{-2}\,du$ write with exponent in numerator

$= 2(-1)u^{-1} + c$ antidifferentiate

$= \frac{-2}{u} + c$ simplify

$= \boxed{\frac{-2}{e^{x} + e^{-x}} + c}$ Substitute back

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x (16) Find the second derivative.

$$g(x) = \sqrt{x} + e^x \ln x$$

$$g'(x) = \frac{1}{2}x^{-1/2} + e^x \cdot \frac{1}{x} + \ln x \cdot e^x$$

$$g''(x) = -\frac{1}{4}x^{-3/2} + e^x\left(\frac{-1}{x^2}\right) + \left(\frac{1}{x}\right)e^x + \ln x \cdot e^x + e^x \cdot \frac{1}{x}$$

$$g''(x) = \frac{-1}{4x^{3/2}} - \frac{e^x}{x^2} + \frac{2e^x}{x} + \ln x \cdot e^x$$

$$\boxed{g''(x) = \frac{-1}{4x^{3/2}} + \frac{(2x-1)e^x}{x^2} + e^x \cdot \ln x}$$

book's version combines where it's easy.

x (17) Find extrema and points of inflection.

$$f(x) = \frac{e^x - e^{-x}}{2}$$

$$f'(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f''(x) = \frac{1}{2}(e^x - e^x)$$

extrema: $f'(x) = 0$

$$\frac{1}{2}(e^x + e^{-x}) = 0$$

$$e^x + \frac{1}{e^x} = 0$$

$$e^x = \frac{-1}{e^x}$$

$$e^{2x} = -1 \quad \text{has no soln.}$$

∴ no critical values

$\boxed{\therefore \text{ no extrema.}}$

inflection: $f''(x) = 0$

$$\frac{1}{2}(e^x - e^x) = 0.$$

$$e^x - \frac{1}{e^x} = 0$$

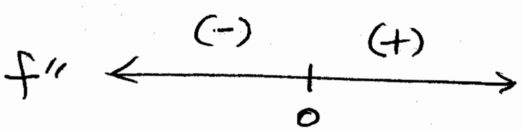
$$e^x = \frac{1}{e^x}$$

$$e^{2x} = 1$$

$$\ln e^{2x} = \ln 1$$

$$2x = 0$$

$$x = 0.$$



$$f(0) = \frac{e^0 - e^0}{2} = 0$$

$\boxed{(0,0) \text{ inflection pt}}$

× (18) Find the area of the region bounded by

$y = e^{-2x} + 2$
$y = 0$
$x = 0$
$x = 2$

$$\text{Area} = \int_0^2 e^{-2x} + 2\,dx$$

$u = -2x \qquad u(0) = 0$ lower
$du = -2\,dx \qquad u(2) = -4$ upper

$$= -\frac{1}{2}\int_0^{-4} e^u\,du + \int_0^2 2\,dx$$

$$= -\frac{1}{2}[e^u]\Big|_0^{-4} + 2x\Big|_0^2$$

$$= -\frac{1}{2}(e^{-4} - e^0) + 2(2-0)$$

$$= \frac{-1}{2e^4} + \frac{1}{2} + 4$$

$$= \frac{9}{2} - \frac{1}{2e^4}$$

$$= \boxed{\frac{9e^4 - 1}{2e^4}}$$

× (19) Approximate the integral using
a) The Trapezoidal Rule
and b) Simpson's Rule
with $n = 12$.

$$\int_0^2 2xe^{-x}\,dx \qquad (\text{fnInt} \approx 1.1880)$$

$0 \quad \frac{1}{6} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{5}{6} \quad 1 \quad \frac{7}{6} \quad \frac{4}{3} \quad \frac{3}{2} \quad \frac{5}{3} \quad \frac{11}{6} \quad 2$

a) $A \approx \frac{b-a}{2(12)}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{11}) + f(x_{12})]$

$$= \frac{2}{24}\left[f(0) + 2\sum_{i=1}^{11} f(x_i) + f(2)\right]$$

$$\approx \boxed{1.1827}$$

b) $\frac{b-a}{3n}\Big[f(0) + 4f(\frac{1}{6}) + 2f(\frac{1}{3}) + 4f(\frac{1}{2}) + 2f(\frac{2}{3}) + 4f(\frac{5}{6}) + 2f(1) + 4f(\frac{7}{6}) + 2f(\frac{4}{3}) + 4f(\frac{3}{2}) + 2f(\frac{5}{3}) + 4f(\frac{11}{6}) + f(2)\Big]$

$\frac{2}{3 \cdot 12} = \frac{1}{18}$

$= \boxed{1.1880}$

(20) $\int \ln(e^{2x-1})\,dx$

$$= \int 2x - 1\,dx$$

$$= \frac{2}{2}x^2 - x + C$$

$$= \boxed{x^2 - x + C}$$

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(21) Solve $\frac{e^x - e^{-x}}{2} = 1$ for x.

X

a) exact solution
b) approx solution by calculating exact
c) approx by GC graphical method.

$\frac{e^x - e^{-x}}{2} = 1$

$e^x - e^{-x} = 2$ clear fractions

$e^x - \frac{1}{e^x} = 2$ write neg exp as positive exp.

$(e^x)^2 - 1 = 2e^x$ clear fractions

$(e^x)^2 - 2e^x - 1 = 0$ notice it's quadratic form $u = e^x$

$u^2 - 2u - 1 = 0$

$u^2 - 2u + 1 = 1 + 1$ doesn't factor

$\left(\frac{-2}{2}\right)^2 = 1$ solve by CTS or QF

$(u-1)^2 = 2$

$u - 1 = \pm\sqrt{2}$

$u = 1 \pm \sqrt{2}$

$e^x = 1 \pm \sqrt{2}$

notice $1+\sqrt{2} \approx 2.4$
$1-\sqrt{2} \approx -.4$
but range of e^x $y > 0$
so no x value gives $e^x = -.4$

$e^x = 1 + \sqrt{2}$

a) $\boxed{x = \ln(1+\sqrt{2})}$ exact

b) approx in GC: ln(1+√(2))
$x \approx .8814$

c) $y_1 = (e^x - e^{-x})/2$
$y_2 = 1$

2nd TRACE = CALC
5. intersect
enter
enter
enter

$x \approx .8814$

Math 250

(22) Solve $\ln 4x - 3\ln x^2 = \ln 2$

combine to single logs on LHS and RHS

$$\ln 4x - \ln(x^2)^3 = \ln 2 \qquad \text{log property}$$

$$\ln(4x) - \ln(x^6) = \ln 2 \qquad \text{algebra}$$

$$\ln\left(\frac{4x}{x^6}\right) = \ln 2 \qquad \text{log property}$$

$$\ln\left(\frac{4}{x^5}\right) = \ln 2 \qquad \text{algebra}$$

Exponentiate both sides $\Leftrightarrow$ set arguments equal.

$$\frac{4}{x^5} = 2$$

$$\frac{4}{2} = x^5$$

$$2 = x^5$$

$$\boxed{x = \sqrt[5]{2}}$$

check: $\ln 4\sqrt[5]{2} - 3\ln(\sqrt[5]{2})^2 = \ln 2$ substitute

$\ln 2^2 \cdot 2^{1/5} - \ln((2^{1/5})^2)^3 = \ln 2$ log properties + same base

$\ln 2^{11/5} - \ln 2^{6/5} = \ln 2$

$\ln\left(\frac{2^{11/5}}{2^{6/5}}\right) = \ln 2$

$\ln 2^{5/5} = \ln 2$ ✓

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$f(x) = e^{-x^2/2}$

23 a) Find all relative extrema, intervals of increase, decrease

b) Find all inflection points, intervals of concavity.

c) Sketch graph

a) $f'(x) = e^{-x^2/2} \cdot -\frac{1}{2}(2x) = -xe^{-x^2/2}$

$f'(x) = 0$

$-xe^{-x^2/2} = 0$

$x = 0 \qquad e^{-x^2/2} \neq 0$

c.v. $x = 0$

f': $+$ on $(-\infty, 0)$, $-$ on $(0, \infty)$

$f'(-1) \approx .6$

$f'(1) \approx -.6$

$f(0) = e^{-0^2/2} = e^0 = 1$

$\boxed{(0,1) \text{ relative max}}$

$\boxed{\begin{array}{l}\text{increasing } (-\infty, 0) \\ \text{decreasing } (0, \infty)\end{array}}$

b) $f''(x) = -x \cdot \frac{d}{dx}\left(e^{-x^2/2}\right) + \frac{d}{dx}(-x) \cdot e^{-x^2/2}$ product rule

$= -x \cdot \left(-xe^{-x^2/2}\right) + -e^{-x^2/2}$

$= x^2 e^{-x^2/2} - e^{-x^2/2}$

$= e^{-x^2/2}(x^2 - 1)$

$= e^{-x^2/2}(x+1)(x-1)$

$f''(x) = 0$ at $x = -1,\ x = 1$

$f''(x)$ undef none

f'': $+$ on $(-\infty, -1)$, $-$ on $(-1, 1)$, $+$ on $(1, \infty)$

$f''(-2) \approx .4$

$f''(0) = -1$

$f''(2) \approx .4$

$f(-1) = e^{-(-1)^2/2} = e^{-1/2}$

$f(1) = e^{-1/2}$

$\boxed{\text{inflection points } \left(-1, e^{-1/2}\right),\ \left(1, e^{-1/2}\right)}$

$\boxed{\begin{array}{l}\text{concave up } (-\infty, -1) \text{ and } (1, \infty) \\ \text{concave down } (-1, 1)\end{array}}$

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cont.

c) x-int: $0 = e^{-x^2/2}$ none

y-int $e^{-0^2/2} = e^0 = 1$ $(0,1)$

symmetry $f(-x) = e^{-(-x)^2/2} = e^{-x^2/2}$ $f(-x) = f(x)$ y-axis

vertical asymptotes: none domain $(-\infty, \infty)$

horizontal asymptotes:

$$\lim_{x\to\infty} f(x)$$ $\left(\lim_{x\to-\infty} f(x) = 0 \text{ also}\right)$

$$= \lim_{x\to\infty} e^{-x^2/2}$$

$$= \lim_{x\to\infty} \frac{1}{e^{x^2/2}}$$

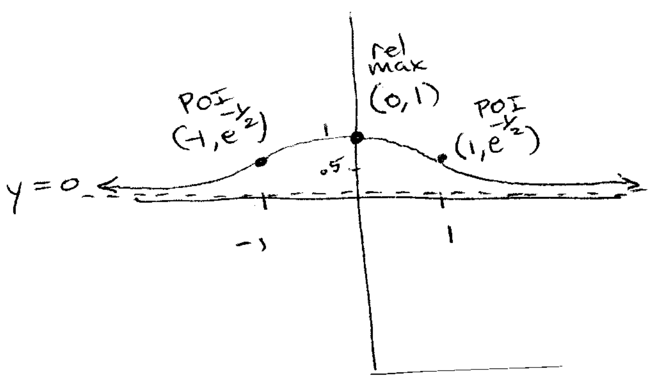
$$= 0$$ $y=0$ approaches x-axis

increasing $(-\infty, 0]$

decreasing $[0, \infty)$

concave up $(-\infty,-1) \cup (1,\infty)$

concave down $(-1,1)$



$(0,1)$ rel max

$\left.\begin{matrix}(1, e^{-1/2}) \\ (-1, e^{-1/2})\end{matrix}\right\}$ inflection

$e^{-1/2} \approx$